VARIOUS METHODS OF SOLVING THE PROBLEM OF MONOTONIC HEATING OF A PLATE

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A solution is obtained for the problem of the heating of a two-layer plate at constant heating rate and for the problem of the monotonic heating of a single-layer plate with allowance for the temperature dependence of the thermophysical properties. The methods used include the integral heat balance method, the small parameter method, and Galerkin's method. The first problem is also solved by an operational method.

The monotonic heating regime is quite widely used for study of the thermophysical properties of materials [1–3]. The exact solution of the problem is given in [4] for the case of constant heating rate and constant thermophysical properties with boundary conditions of the third kind. When the monotonic heating method is used to determine thermal conductivity, the test material is brought into contact with a certain standard [3], which complicates the boundary conditions. We first find the solution of the latter problem for constant λ and c by the exact and approximate methods.

In the general case, the equation of the plane one-dimensional problem has the form:

$$c(\vartheta)\frac{\partial \vartheta}{\partial \tau}-\frac{\partial}{\partial x}\left[\lambda(\vartheta)\frac{\partial \vartheta}{\partial x}\right]=0. \tag{1}$$

Assuming that the standard plate is made of metal with high thermal conductivity and has the same temperature over its entire thickness, we can write the initial and boundary conditions in the form

$$\vartheta(x,\ 0)=0, \tag{2}$$

$$\frac{\partial \vartheta}{\partial x}(0,\ \tau)=\frac{c'h'}{\lambda}\frac{\partial \vartheta}{\partial \tau}(0,\ \tau);\quad \frac{\partial \vartheta}{\partial \tau}(h,\ \tau)=b. \tag{3}$$

Using the Laplace transformation, we obtain the solution for the transform with constant λ and c and the initial condition (2) in the following form:

$$\vartheta_L(x,\ s)=A\,\mathrm{ch}\sqrt{\frac{s}{a}}\,x+B\,\mathrm{sh}\sqrt{\frac{s}{a}}\,x. \tag{4}$$

Finding the constants A and B from the boundary conditions, we have

$$\vartheta_L(x,\ s)=b\,\frac{\mathrm{ch}\sqrt{\frac{s}{a}}\,x+\frac{h}{\gamma}\sqrt{\frac{s}{a}}\,\mathrm{sh}\sqrt{\frac{s}{a}}\,x}{s^2\left(\mathrm{ch}\sqrt{\frac{s}{a}}\,h+\frac{h}{\gamma}\sqrt{\frac{s}{a}}\,\mathrm{sh}\sqrt{\frac{s}{a}}\,h\right)}. \tag{5}$$

Going over to the inverse transform, we obtain

$$\vartheta=b\tau-\frac{bh^2}{a}\left[\frac{1}{\gamma}\left(1-\frac{x}{h}\right)+\frac{1}{2}\left(1-\frac{x^2}{h^2}\right)\right]+$$

$$+\frac{bh^2}{a}\sum_{n=1}^{\infty}\frac{A_n}{\mu_n^2}\left(\cos\mu_n\frac{x}{h}-\frac{\mu_n}{\gamma}\sin\mu_n\frac{x}{h}\right)\exp(-\mu_n^2\,\mathrm{Fo}), \tag{6}$$

where

$$A_n=\frac{2\sin\mu_n}{\mu_n+\sin\mu_n\cos\mu_n}$$

and the μ_n are found from the characteristic equation

$$\operatorname{ctg}\mu_n = \frac{\mu_n}{\gamma}. \tag{7}$$

At $\gamma = \infty$ this equation reduces [4] to the known equation for a single plate.

The solution of Eq. (1) by the integral heat balance method is based on its integration over the thickness of the thermal layer y

$$F(y) - \lambda(\vartheta_y)\frac{\partial\vartheta_y}{\partial x} + \lambda(\vartheta_0)\frac{\partial\vartheta(0)}{\partial x} = 0, \tag{8}$$

where

$$F(y) = \int_0^y c[\vartheta(y, x)]\frac{\partial\vartheta(y, x)}{\partial\tau}\,dx. \tag{9}$$

In accordance with the procedure of [5], we divide the solution into two stages. The boundary conditions for the first stage (propagation of the initial perturbation over the entire thickness of the body) are written as follows:

$$\vartheta(x, 0) = 0;\ \vartheta(\delta, \tau) = 0;\ \frac{\partial\vartheta}{\partial x}(\delta, \tau) = 0;\ \frac{\partial\vartheta}{\partial\tau}(h, \tau) = b. \tag{10}$$

The first stage is not affected by the standard plate. Approximation of the temperature profile by a quadratic function gives, in accordance with (10),

$$\vartheta = b\tau\left(\frac{x-\delta}{h-\delta}\right)^2. \tag{11}$$

Substituting (11) into (8), we obtain the differential equation

$$\tau(h-\delta)\frac{d\delta}{d\tau} - (h-\delta)^2 + 6a\tau = 0, \tag{12}$$

whose solution has the form

$$\delta = h - 2\sqrt{a\tau}. \tag{13}$$

At the end of the first stage $\delta_1 = 0$, whence

$$\tau_1 = \frac{h^2}{4a} \tag{14}$$

and $Fo_1 = 0.25$.

In the second stage, we supplement the boundary conditions (3) with the initial conditions

$$\vartheta(0, \tau_1) = 0;\ \frac{\partial\vartheta}{\partial x}(0, \tau_1) = 0. \tag{15}$$

The temperature profile in the second stage

$$\vartheta = b\tau - A_1(h-x) - A_2(h^2 - x^2). \tag{16}$$

Using the boundary conditions and Eq. (8), for finding the coefficients in (16) we obtain a system of two equations

$$\frac{dA_1}{d\,Fo} = \frac{3\gamma}{h}A_1 - 12A_2 + 3\frac{b}{a}, \tag{17}$$

$$\frac{dA_2}{d\,Fo} = -4\gamma A_1 + 12A_2h - 2\frac{bh}{a}. \tag{18}$$

Hence, we find

$$A_1 = \frac{bh}{a\gamma} - \beta_1 \exp(\alpha_1 \mathrm{Fo}) - \beta_2 \exp(\alpha_2 \mathrm{Fo}), \tag{19}$$

$$A_2 = \frac{b}{2a} - \beta_1 \frac{\alpha_1 + 4\gamma}{12h} \exp(\alpha_1 \mathrm{Fo}) - \beta_2 \frac{3\gamma}{h(\alpha_2 + 12)} \exp(\alpha_2 \mathrm{Fo}), \tag{20}$$

$$\alpha_{1,2} = -(2\gamma + 6) \pm \sqrt{4\gamma^2 + 12\gamma + 36}. \tag{21}$$

The coefficients β_1 and β_2 are found using the initial conditions for the second stage

$$\beta_1 = \frac{bh}{a} \frac{3\alpha_2}{(\alpha_1 + 4\gamma)(\alpha_2 + 12) - 36\gamma} \exp\left(-\frac{\alpha_1}{4}\right), \tag{22}$$

$$\beta_2 = \frac{bh}{a} \left[\frac{1}{\gamma} - \frac{3\alpha_2}{(\alpha_1 + 4\gamma)(\alpha_2 + 12) - 36\gamma}\right] \exp\left(-\frac{\alpha_2}{4}\right). \tag{23}$$

For the temperature profile we finally obtain

$$\vartheta = b\tau - \frac{bh^2}{a}\left[\frac{1}{\gamma}\left(1 - \frac{x}{h}\right) + \frac{1}{2}\left(1 - \frac{x^2}{h^2}\right)\right] + \beta_1\left[h - x + \frac{\alpha_1 + 4\gamma}{12h}(h^2 - x^2)\right]\exp(\alpha_1 \mathrm{Fo}) + \beta_2\left[h - x + \frac{3\gamma}{h(\alpha_2 + 12)}(h^2 - x^2)\right]\exp(\alpha_2 \mathrm{Fo}). \tag{24}$$

In the special case $\gamma = \infty$ (a single plate of thickness h adiabatically insulated in the plane $x = 0$ or a plate of thickness 2k heated on both sides) we have $\alpha_1 = -3$ and $\alpha_2 = -\infty$, whence

$$\vartheta = b\tau - \frac{bh^2}{2a}\left(1 - \frac{x^2}{h^2}\right)\left[1 - \frac{1}{2}\exp\left(-3\mathrm{Fo} + \frac{3}{4}\right)\right]. \tag{25}$$

We now employ Galerkin's method [6] to solve this problem. In accordance with this method, the solution of the equation

$$L(u) = 0, \tag{26}$$

where L is some differential operator, is obtained in the form

$$\bar{u}(x, y) = \sum_{i=1}^{n} c_i \varphi_i(x, y), \tag{27}$$

where $\varphi_i(x, y)$ ($i = 1, 2, \ldots, n$) is some system of preselected functions satisfying the boundary conditions of Eqs. (26). For finding the coefficients c_i the method gives a system of algebraic equations obtained by the integration of:

$$\iint_D L[\bar{u}(x, y)]\, \varphi_i(x, y)\, dxdy = \iint_D L\left[\sum_{j=1}^{n} c_j \varphi_j(x, y)\right] \varphi_i(x, y)\, dxdy = 0. \tag{28}$$

The selection of simple functions satisfying all the boundary conditions of the problem, especially the first of conditions (3), presents considerable difficulties. Therefore, we confine ourselves to finding an approximate one-term solution for $\gamma = \infty$. It is convenient to seek the solution in the form

$$\bar{\vartheta} = \bar{\vartheta}_\infty + \bar{\vartheta}_i, \tag{29}$$

where $\bar{\vartheta}_\infty$ is the temperature after establishment of the quasi-stationary regime,

$$\bar{\vartheta}_\infty = b\tau - \frac{bh^2}{2a}\left(1 - \frac{x^2}{h^2}\right), \tag{30}$$

$$\bar{\vartheta}_i = \frac{bh^2}{2a}\left(1 - \frac{x^2}{h^2}\right)\exp(-B\,\mathrm{Fo}). \tag{31}$$

In our case, Eq. (26) has the form of (1). Since $L(\bar{\vartheta}_\infty) \equiv 0$, we can substitute $\bar{\vartheta}_i$ for $\bar{\vartheta}$ into (28). As a result we have

$$\int\limits_{x=0}^{h}\int\limits_{\mathrm{Fo}=0}^{\infty}\left[1 - \frac{x^2}{h^2} - B\left(1 - \frac{x^2}{h^2}\right)^2\right]\exp(-B\,\mathrm{Fo})\,dx\,d\,\mathrm{Fo} = 0, \tag{32}$$

whence B = 2.5.

For the region of small Fo, a one-term approximation of type (29)–(31) is no longer satisfactory. In this region, we use Galerkin's method to find an approximate solution in the form, analogous to (11):

$$\bar{\vartheta} = b\tau\left(1 - \frac{1 - \frac{x}{h}}{\sqrt{\varepsilon\tau}}\right)^2. \tag{33}$$

Clearly, an equation of this type is applicable in the region $0 < 1 - x/h < (\varepsilon\tau)^{1/2}$. Substituting (33) into (1) and integrating $L(\bar{\vartheta})\bar{\vartheta}$ with respect to x from 1 to $1 + (\varepsilon\tau)^{1/2}$, we obtain

$$\varepsilon = 4a. \tag{34}$$

This value is equal to that obtained by the integral heat balance method.

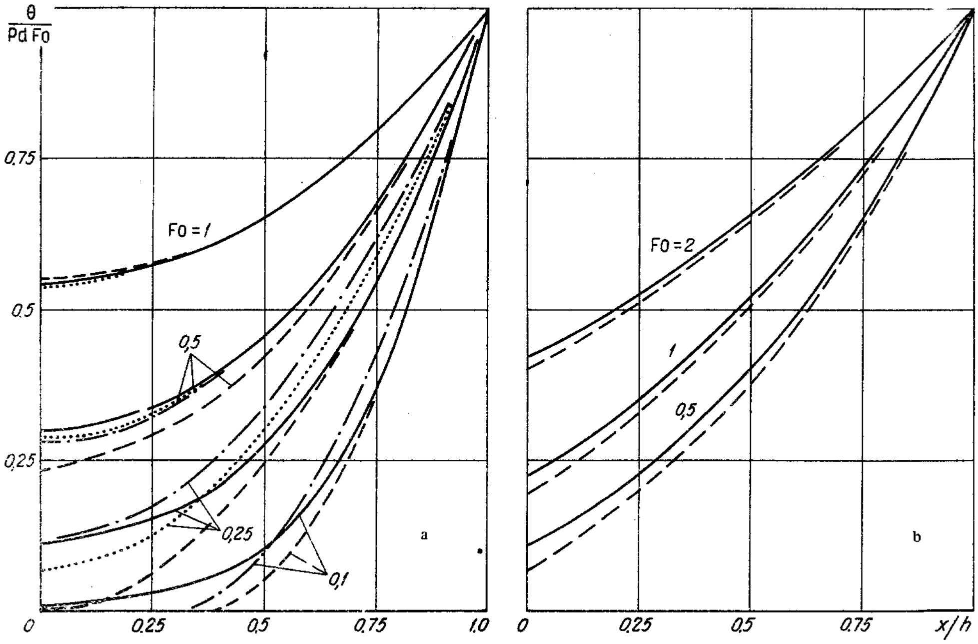

Fig. 1. Comparison of exact and approximate solutions for the heating of a plate with constant thermophysical properties ((a) $\gamma = \infty$; (b) $\gamma = 1$). Solid lines—exact solution [Eq. (6)]; dashed lines—integral heat balance method [Eqs. (11) and (24)]; dotted lines—Galerkin's method [Eq. (31)]; dot-dash lines—Biot's variational method [7].

In [7] an approximate solution of the problem for $\gamma = \infty$ was obtained by Biot's variational method. Comparison of the approximate solutions obtained by the three methods with the exact solutions at $\gamma = \infty$ is given in Fig. 1a. The Galerkin one-term approximation gives the least discrepancy. As may be seen from Fig. 1b, the integral heat balance method gives a satisfactory approximation at small γ.

We find a solution of the problem for the region of the quasi-stationary regime at $\gamma = \infty$ taking into account the temperature dependence of the thermophysical properties. Equation (1) can be written in the form

$$\frac{\partial^2 \vartheta}{\partial x^2} + k_\lambda \left(\frac{\partial \vartheta}{\partial x}\right)^2 - \frac{1}{a} \frac{\partial \vartheta}{\partial \tau} = 0, \tag{35}$$

where $k_\lambda = (1/\lambda)(\partial\lambda/\partial\vartheta)$ is the relative temperature coefficient of thermal conductivity. In [8] an approximate solution of Eq. (35) is given for the region of the quasi-stationary regime by the small parameter method. The temperature dependence of the thermophysical parameters is approximated by the linear functions

$$\lambda = \lambda_0 (1 + k_\lambda \vartheta) \text{ and } a = a_0 (1 + k_a \vartheta) \cong \frac{a_0}{1 - k_a \vartheta}. \tag{36}$$

When $k_a\vartheta \ll 1$ Eq. (35) takes the form

$$\frac{\partial^2 \vartheta}{\partial x^2} + k_\lambda \left(\frac{\partial \vartheta}{\partial x}\right)^2 - \frac{1 - k_a \vartheta}{a_0} \frac{\partial \vartheta}{\partial \tau} = 0. \tag{37}$$

The rate of increase of the temperature of the different layers of the plate can also be represented by a linear function

$$b = b_0 (1 + k_{b,x} \vartheta).$$

When $k_{b,x}\vartheta \ll 1$

$$\frac{d^2 \vartheta}{dx^2} + k_\lambda \left(\frac{d \vartheta}{dx}\right)^2 + (k_a - k_{b,x}) \frac{b_0}{a_0} \vartheta - \frac{b_0}{a_0} = 0. \tag{38}$$

In [8] an approximate solution of this nonlinear equation was obtained:

$$\Delta\vartheta = \vartheta(x, \tau) - \vartheta(0, \tau) = \frac{b_0 x^2}{2a_0} \times$$
$$\times \left[1 - \frac{b_0}{12a_0} (2k_\lambda + k_a - k_{b,x}) x^2\right]. \tag{39}$$

Let us find a more accurate solution of Eq. (38). By means of the substitution $z = (d\vartheta/dx)^2$, Eq. 38 is transformed into the linear equation

$$\frac{dz}{d\vartheta} + 2k_\lambda z + 2\frac{b_0}{a_0} [(k_a - k_{b,x})\vartheta - 1] = 0, \tag{40}$$

solving which, with allowance for the boundary conditions

$$\vartheta(0, \tau) = \vartheta_0; \quad \frac{d\vartheta}{dx}(0, \tau) = 0, \tag{41}$$

we obtain

$$z = \frac{b_0}{a_0 k_\lambda}\left(1 + \frac{k_a - k_{b,x}}{2k_\lambda}\right)\{1 - \exp[-2k_\lambda(\vartheta - \vartheta_0)]\} -$$
$$- \frac{b_0 (k_a - k_{b,x})}{a_0 k_\lambda} \{\vartheta - \vartheta_0 \exp[-2k_\lambda(\vartheta - \vartheta_0)]\}. \tag{42}$$

Expanding $\exp[-2k_\lambda(\vartheta - \vartheta_0)]$, confining ourselves to the first three terms of the series (the corresponding error does not exceed 3% at $2k_\lambda(\vartheta - \vartheta_0) < 0.5$) and carrying out the integration, we obtain a formula determining the temperature of the free wall of the plate ϑ_0

$$b_0 \tau = \vartheta_0 + \frac{2 - \frac{k_a - k_{b,x}}{k_\lambda}\vartheta_0}{2k_\lambda + k_a - k_{b,x} - 2k_\lambda (k_a - k_{b,x})\vartheta_0} \times$$
$$\times \sin^2 \frac{h}{2} \sqrt{\frac{b_0}{a_0} [2k_\lambda + k_a - k_{b,x} - 2k_\lambda (k_a - k_{b,x})\vartheta_0]}. \tag{43}$$

When $\vartheta_0 = 0$, we obtain a simple equation for the temperature profile

$$\Delta\vartheta = \frac{1}{2k_\lambda + k_a - k_{b,x}} \left[1 - \cos x \sqrt{\frac{b_0}{a_0}(2k_\lambda + k_a - k_{b,x})}\right]. \qquad (44)$$

If we confine ourselves to the first three terms of the expansion of the cosine, we obtain Eq. (39). In accordance with our assumptions, Eq. (44) describes the actual temperature field with an error of not more than 1%, if the argument of the cosine does not exceed $\pi/2$. In this region, Eq. (39) gives very similar values, but is somewhat less convenient for solving the inverse problem of determining the thermal diffusivity from temperature field measurements. In accordance with (44), the thermal diffusivity of the material at the temperature of the boundary surface $x = 0$

$$a_0 = \frac{b_0 x^2 (2k_\lambda + k_a - k_{b,x})}{\{\arccos [1 - \Delta\vartheta (2k_\lambda + k_a - k_{b,x})]\}^2}. \qquad (45)$$

In [9] the effect of the temperature dependence of the thermophysical properties is taken into account in a more complicated way using the solution for multilayer systems. In the example considered in that paper $b_0 = 1200$ deg/hr; $h = 0.05$ mm; $a_0 = 0.015$ m^2/hr; $k_\lambda = 0.001$ 1/deg; and $k_a = 0.0005$ 1/deg. As may be seen from Fig. 2, the two methods give almost identical results.

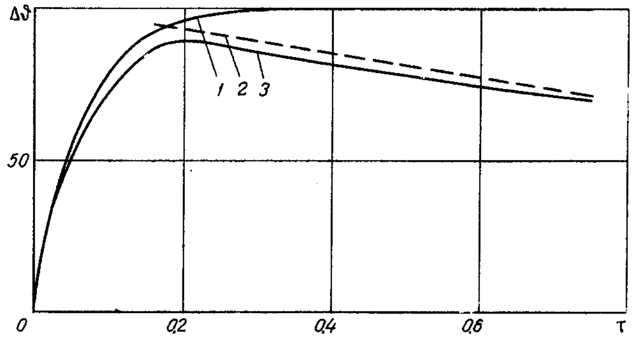

Fig. 2. Effect of temperature dependence of the thermophysical properties on the heating of a plate ($\Delta\vartheta$, deg; τ, hr): 1) Eq. (6); 2) [9]; 3) Eqs. (44) and (46).

An approximate solution taking the temperature dependence of the thermophysical properties into account can be found for the region of the nonstationary regime by the Galerkin method. We find the solution in the form of the one-term approximation that gave good agreement with the exact solution for constant thermophysical parameters. In accordance with (39)

$$\bar{\vartheta}_{\text{н}} = \frac{b_0 h^2}{2a_0}\left[1 - \frac{x^2}{h^2} - D\left(1 - \frac{x^4}{h^4}\right)\right] \exp(-C\,\text{Fo}), \qquad (46)$$

where

$$D = \frac{b_0 h^2}{12 a_0}(2k_\lambda + k_a - k_{b,x}).$$

In this case the differential operator is determined from Eq. (37). The solution has the form

$$C = \frac{6M - 4Nk_\lambda \dfrac{b_0 h^2}{a_0}}{3P - Qk_a \dfrac{b_0 h^2}{a_0}}, \qquad (47)$$

where

$$M = \frac{2}{3} - \frac{86}{35} D + 2D^2;$$

$$N = \frac{2}{15} - \frac{44}{105} D + \frac{152}{315} D^2 - \frac{16}{77} D^3;$$

$$P = \frac{8}{15} - \frac{128}{105} D + \frac{32}{45} D^2;$$

$$Q = \frac{16}{35} - \frac{2}{3} D + \frac{142}{165} D^2 - \frac{128}{195} D^3.$$

If $D \ll 1$,

$$C = \frac{5 - \frac{2}{3} k_\lambda \frac{b_0 h^2}{a_0}}{2 - \frac{4}{7} k_a \frac{b_0 h^2}{a_0}} .$$

Calculations based on Eq. (47) give C = 2.36 instead of the value of 2.50 for constant thermophysical properties. The initial section of curve 3 in Fig. 2 was calculated from equation (46) with the value found for C.

NOTATION

a is the thermal diffusivity
b is the heating rate
c is the specific heat
h is the thickness of the plate
T is the temperature
T_0 is the temperature at the initial instant
x is the coordinate
δ is the thickness of the thermal layer
λ is the thermal conductivity
τ is the time
$\vartheta = T - T_0$
$\theta = (T - T_0)/T_0$
$Fo = a\tau/h^2$ is the Fourier number
$Pd = bh^2/aT_0$ is the Predvoditelev number
$\gamma = ch/c'h'$
A prime refers to the standard plate

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